Let us introduce the "fusion rule" for counting dimensions of conformal blocks:
Take $n+1$ distinct points $p_1, \dots, p_n, p_{n+1}$ so that $p_{n+1} = \infty \longrightarrow$ associated integrable highest weight modules: $H_{\lambda_1}, \dots, H_{\lambda_n}, H^*_{\lambda_{n+1}}$ of level K.

$\uparrow$ dual module

Define conformal blocks

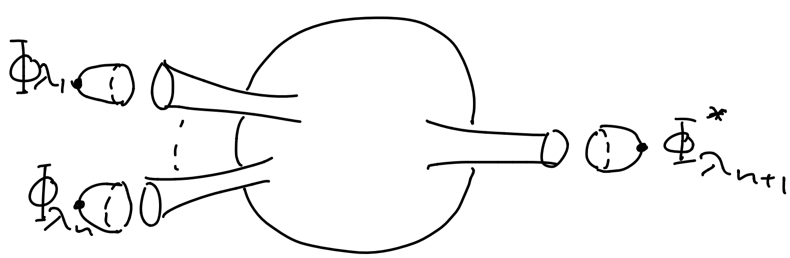
$$\mathcal{H}(p_1, \dots, p_n, p_{n+1}; \lambda_1, \dots, \lambda_n, \lambda^*_{n+1})$$

$$\equiv \mathrm{Hom}_{\mathfrak{g}(p_1, \dots, p_n, p_{n+1})}\left(\left(\bigotimes_{j=1}^{n} H_{\lambda_j}\right) \otimes H^*_{\lambda_{n+1}}, \mathbb{C}\right)$$

$$\downarrow$$

$$\mathrm{Hom}_{\mathfrak{g}}\left(\left(\bigotimes_{j=1}^{n} V_{\lambda_j}\right) \otimes V^*_{\lambda_{n+1}}, \mathbb{C}\right)$$

with $\dim \mathcal{H} = N^{\lambda_{n+1}}_{\lambda_1 \cdots \lambda_n}$.



In the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$: $N^{\lambda_{n+1}}_{\lambda_1 \cdots \lambda_n} = N_{\lambda_1 \lambda_2 \cdots \lambda_{n+1}}$ since dual rep. is equiv. to original one.

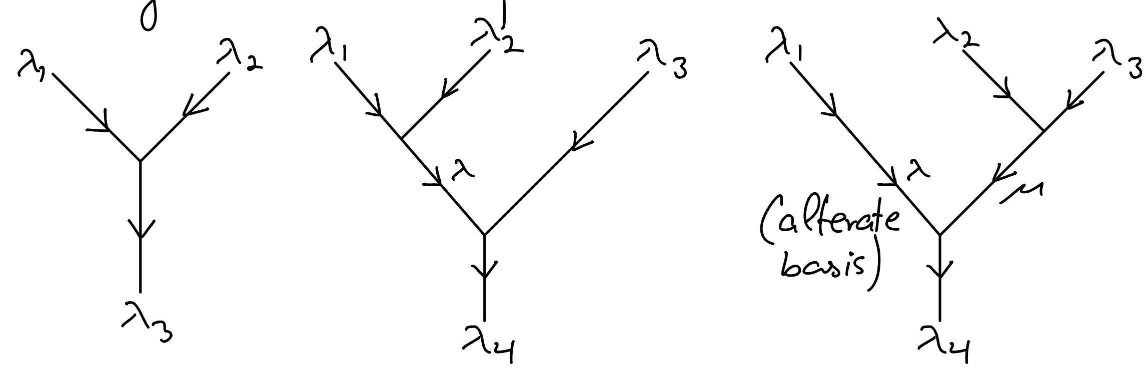
In the case $\underline{n=3}$: $\mathcal{H} \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}, V_{\lambda_4})$

We have the following composition of projection maps:

$$p^{\lambda}_{\lambda_1, \lambda_2} \otimes \mathrm{id}_{V_{\lambda_3}} : (V_{\lambda_1} \otimes V_{\lambda_2}) \otimes V_{\lambda_3} \longrightarrow V_\lambda \otimes V_{\lambda_3}$$

$$p^{\lambda_4}_{\lambda \lambda_3} : V_\lambda \otimes V_{\lambda_3} \longrightarrow V_{\lambda_4}$$

$\longrightarrow$ gives basis for $\mathcal{H}$ "labelled trees"



Denote the above trees as Γ_n, where

1. Each vertex of Γ_n has valency 3 or 1
2. The number of vertices of Γ_n with valency 1 is equal to $n+1$ (external edges $\gamma_i, 1 \le i \le n+1$)
3. Introduce an "admissible labelling" f for Γ_n (each vertex satisfies quantum CGC)

$\longrightarrow$ gives basis of conformal blocks.

§ 5. KZ equation

Definition:

The "configuration space" of n ordered distinct points on $\mathbb{CP}^1$ is

$$\mathrm{Conf}_n(\mathbb{CP}^1) = \{(p_1, \dots, p_n) \in (\mathbb{CP}^1)^n \mid p_i \neq p_j \text{ if } i \neq j\}$$

Now choose level k highest weights $\lambda_1, \dots, \lambda_n$ and consider the space

$$\mathcal{E}_{\lambda_1, \dots, \lambda_n} = \bigcup_{(p_1, \dots, p_n) \in \mathrm{Conf}_n(\mathbb{CP}^1)} \mathcal{H}(p_1, \dots, p_n ; \lambda_1, \dots, \lambda_n)$$

as a vector bundle on $\mathcal{E}_{\lambda_1, \dots, \lambda_n}$

$\longrightarrow$ consider trivial vector bundle

$$E = \mathrm{Conf}_n(\mathbb{CP}^1) \times \mathrm{Hom}_{\mathbb{C}}\left(\bigotimes_{j=1}^{n} H_{\lambda_j}, \mathbb{C}\right)$$

and $\mathcal{E}_{\lambda_1, \dots, \lambda_n} \subset E$ as subset.

For $j = 1, \dots, n$ consider hyperplanes D_j of $(\mathbb{CP}^1)^{n+1}$ defined by $z_{n+1} = z_j$ where $(z_1, \dots, z_{n+1}) \in (\mathbb{CP}^1)^{n+1}$. Let U be open set of $\mathrm{Conf}(\mathbb{CP}^1)$ and denote by

$\mathcal{M}_{D_1 \cdots D_n}(U)$ set of meromorphic functs. !
with poles of any order at most at $D_1, \dots, D_n$ $\longrightarrow$ $\mathfrak{g} \otimes \mathcal{M}_{D_1, \dots, D_n}(U)$ has structure of Lie algebra.

$f \in \mathfrak{g} \otimes \mathcal{M}_{D_1 \cdots D_n}(U)$ has Laurent expansion along D_j, $1 \le j \le n$, of the form

$$f_{D_j}(t_j) = \sum_{m=-N}^{\infty} a_m(z_1, \dots, z_n)\, t_j^m . \quad (*)$$

where $t_j = z_{n+1} - z_j$ and $a_m(z_1, \dots, z_n)$ is a holomorphic function in $z_1, \dots, z_n$ with values in $\mathfrak{g}$. Denote by $\mathcal{O}(U)$ set of hol. functions on U. Then $(*)$ gives a map

$$\tau_j : \mathfrak{g} \otimes \mathcal{M}_{D_1 \cdots D_n}(U) \longrightarrow \mathfrak{g} \otimes \mathcal{O}(U) \otimes \mathbb{C}((t_j))$$

determined by $\tau_j(f) = f_{D_j}(t_j)$.

$\longrightarrow$ get action of Lie algebra $\mathfrak{g} \otimes \mathcal{M}_{D_1 \cdots D_n}(U)$ on $\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}$ depending on $(p_1, \dots, p_n) \in U$.

Define $\mathcal{E}_{\lambda_1\cdots\lambda_n}(U)$ to be set of smooth sections

$$\Psi : U \longrightarrow E$$

satisfying

$$\sum_{j=1}^{n} [\Psi(p_1,\dots,p_n)](\xi_1,\dots,\tau_j(f)\xi_j,\dots,\xi_n)=0$$

for any $f \in \mathfrak{g}\otimes \mathcal{M}_{D_1\cdots D_n}(U)$ and $\xi_j \in H_{\lambda_j}$, $1 \le j \le n$ at any $(p_1,\dots,p_n)\in U$. Note

$$\Psi(p_1,\dots,p_n)\in \mathcal{H}(\vec{p};\vec{\lambda})$$

Have to show: $\mathcal{E}_{\lambda_1\cdots\lambda_n}$ has structure of vector bundle with flat connection on open subset

$$\mathrm{Conf}_n(\mathbb{C}) = \{(z_1,\dots,z_n)\in\mathbb{C}^n \mid z_i\neq z_j \text{ if } i\neq j\}$$

of $\mathrm{Conf}_n(\mathbb{CP}^1)$. To this end, define multi-linear map $X^{(j)}\Psi : H_{\lambda_1}\otimes\cdots\otimes H_{\lambda_n}\to\mathbb{C}$ by

$$[X^{(j)}\Psi](\xi_1,\dots,\xi_n)=\Psi(\xi_1,\dots,X\xi_j,\dots,\xi_n),$$

where $\xi_1\in H_{\lambda_1},\dots,\xi_n\in H_{\lambda_n}$ and $X\in\hat{\mathfrak{g}}$ acts on jth component.

Proposition 1:

If Ψ is a smooth section of $\mathcal{E}_{\lambda_1,\dots,\lambda_n}$ over open subset $U \subset \mathrm{Conf}_n(\mathbb{C})$, then

$$\frac{\partial \Psi}{\partial z_j} - L_{-1}^{(j)} \Psi$$

is smooth section of $\mathcal{E}_{\lambda_1,\dots,\lambda_n}$ over U for all $1 \le j \le n$.

Proof:

For $f \in \mathfrak{g} \otimes \mathcal{M}_{D_1 \dots D_n}(U)$ get Laurent expansion

$$\tau_j(f) = f_{D_j}(t_j) = \sum_{m=-N}^{\infty} a_m(z_1,\dots,z_n)\, t_j^n$$

along D_j, where $t_j = z_{n+1} - z_j$.

$\Rightarrow f_{z_j} = \frac{\partial f}{\partial z_j} \in \mathfrak{g} \otimes \mathcal{M}_{D_1 \dots D_n}(U)$

with Laurent expansion

$$\tau_j(f_{z_j}) = \sum_{m=-N}^{\infty} \left(\frac{\partial a_m}{\partial z_j} t_j^m - a_m m t_j^{m-1} \right)$$

Define operator $\partial_j : \mathfrak{g} \otimes \mathcal{O}(U) \to \mathfrak{g} \otimes \mathcal{O}(U)$, $1 \le j \le n$ by $\partial_j(h) = h_{z_j}$ and extend to action on $\mathfrak{g} \otimes \mathcal{O}(U) \otimes \mathbb{C}((t_j))$ with trivial action on $\mathbb{C}((t_j))$.

$$\longrightarrow \tau_j(f_{z_j}) = \partial_j \tau_j(f) - \frac{\partial}{\partial t_j}\tau_j(f)$$

By Prop. 6 of §2 we have

$$\frac{\partial}{\partial t_j}\tau_j(f) = -[L_{-1}, \tau_j(f)]$$

$$\Rightarrow \tau_j(f_{z_j}) = \partial_j \tau_j(f) + [L_{-1}, \tau_j(f)]$$

Have to show that:

$$\sum_{i=1}^{n}\left(\frac{\partial \Psi}{\partial z_j} - L_{-1}^{(j)}\Psi\right)(\xi_1, \dots, \tau_i(f)\xi_i, \dots, \xi_n) = 0$$

We have

$$\frac{\partial}{\partial z_j}\left[\Psi(\xi_1, \dots, \tau_i(f)\xi_i, \dots, \xi_n)\right]$$

$$= \left(\frac{\partial \Psi}{\partial z_j}\right)(\xi_1, \dots, \tau_i(f)\xi_i, \dots, \xi_n) + \Psi(\xi_1, \dots, \partial_j \tau_i(f)\xi_i, \dots \xi_n)$$

Altogether we thus obtain:

$$\sum_{i=1}^{n}\left(\frac{\partial \Psi}{\partial z_j} - L_{-1}^{(j)}\Psi\right)(\xi_1, \dots, \tau_i(f)\xi_i, \dots, \xi_n)$$

$$= \sum_{i=1}^{n}\left(\frac{\partial}{\partial z_j}\left[\Psi(\xi_1, \dots, \tau_i(f)\xi_i, \dots, \xi_n)\right] - \Psi(\xi_1, \dots, \tau_i(f_{z_j})\xi_i, \dots, \xi_n)\right)$$

$= 0$ by definition of Ψ $\square$

Introduce the following linear operator:

$$\nabla_{\frac{\partial}{\partial z_j}} : \mathcal{E}_{\lambda_1 \cdots \lambda_n}(U) \longrightarrow \mathcal{E}_{\lambda_1 \cdots \lambda_n}(U)$$

defined by

$$\nabla_{\frac{\partial}{\partial z_j}} \Psi = \frac{\partial \Psi}{\partial z_j} - L_{-1}^{(j)} \Psi$$

<u>Theorem 1:</u>

The family of conformal blocks $\mathcal{E}_{\lambda_1 \cdots \lambda_n}$ over the configuration space $\mathrm{Conf}_n(\mathbb{C})$ has the structure of a vector bundle with a flat connection.

<u>Proof:</u>

We set $M = \mathrm{Conf}_n(\mathbb{C})$ and consider $E = M \times F$ with $F = \mathrm{Hom}_{\mathbb{C}}\left(\bigotimes_{j=1}^{n} H_{\lambda_j}, \mathbb{C}\right)$

Introduce connection

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^* M_{\mathbb{C}} \otimes E)$$

given by $\nabla \Psi = d\Psi - \underbrace{\sum_{i=1}^{n} L_{-1}^{(i)} \Psi \, dz_i}_{=: \omega \Psi}$

$\rightarrow d\omega = 0$ ($L_{-1}^{(i)}$ indep. of z)

$\omega \wedge \omega = 0$ $\left([L_{-1}^{(i)}, L_{-1}^{(j)}] = 0, \; 1 \leq i \leq j \leq n \right)$

$\rightarrow \nabla$ is flat connection $\square$